Calculus 2 Final Exam

6 April 2020,

18:45–22:15 (UTC+02:00)

university of groningen

INSTRUCTIONS – READ CAREFULLY

The exam consists of 6 problems. You have 210 minutes to upload your solutions to your personal Dropbox folder on the Nestor page of Calculus 2. You can achieve 100 points, which includes a bonus of 10 points. You may freely consult the literature, your personal notes, or the available course documents. You may *not* consult other people in any way; you must arrive at your answers independently.

The exact problems you need to solve depend on the seven digits in your student number s$n_7n_6n_5n_4n_3n_2n_1$. You need to substitute these digits in the formulation of each problem before solving it.

Example. Your student number is s7654321 and you are asked to evaluate

$$\int_{3+n_1}^{9+n_2} (-1)^{n_3} x^{10} + x^{2n_4} + n_5 x + \cos(n_6\pi)\ dx.$$

That means that you are supposed to evaluate

$$\int_{3+1}^{9+2} (-1)^3 x^{10} + x^{2\cdot 4} + 5x + \cos(6\pi)\ dx = \int_4^{11} -x^{10} + x^8 + 5x + 1\ dx.$$

THE PROBLEMS

1. **[5+5+5=15 Points]**

   Let the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined as for $(x,y) = (0,0)$ as $f(x,y) = 0$ and for $(x,y) \neq (0,0)$ as

$$f(x,y) = \begin{cases} \dfrac{(n_1+1)x^5 + (-1)^{n_2}x^2y^3 - (n_1+1)y^5}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

   (a) Is f continuous at $(x,y) = (0,0)$? Justify your answer.

   (b) Use the definition of directional derivatives to determine the directional derivative $D_{\boldsymbol{u}}f(0,0)$ for unit vectors $\boldsymbol{u} = (v,w) \in \mathbb{R}^2$.

   (c) Compute the linearization of f at $(x,y) = (0,0)$ and use the definition of differentiability to decide whether f is differentiable at $(x,y) = (0,0)$.

2. **[4+4+7=15 Points]** Consider the curve parametrized by $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^3$ with

$$\mathbf{r}(t) = (-1)^{n_2}\sin((n_1+1)t)\,\mathbf{i} + t\,\mathbf{j} - (-1)^{n_3}\cos((n_1+1)t)\,\mathbf{k}.$$

   (a) Determine the length of the curve and its parametrization by arclength s.

   (b) At each point on the curve, determine the unit tangent vector $\mathbf{T}$ and the curvature κ.

   (c) Let $\mathbf{N}$ be the unit vector with direction $\frac{d}{ds}\mathbf{T}$ and let $\mathbf{B}$ be the unit vector defined as $\mathbf{B} = \mathbf{T}\times\mathbf{N}$. Compute $\mathbf{B}$ and show that $\frac{d}{ds}\mathbf{B} = -\tau\mathbf{N}$ for some $\tau \in \mathbb{R}$. Determine τ.

3. **[3+6+6=15 Points]** Let S be the ellipsoid in $\mathbb{R}^3$ defined by

$$(n_1+1)x^2+(n_2+1)y^2+(n_3+1)z^2 = n_1+n_2+n_3+3$$

which contains the point $(x_0, y_0, z_0) = ((-1)^{n_1}, (-1)^{n_2}, (-1)^{n_3})$.

(a) Compute the tangent plane of S at the point (x_0, y_0, z_0).

(b) Show that near the point (x_0, y_0, z_0) the ellipsoid S is locally given as the graph of a function over the (x, y) plane, i.e. there is a function $f : (x, y) \mapsto f(x, y)$ such that near (x_0, y_0, z_0) the ellipsoid is locally given by $z = f(x, y)$. Compute the partial derivatives f_x and f_y at (x_0, y_0) and show that the graph of the linearization of f at (x_0, y_0) agrees with the tangent plane found in part (a).

(c) Use the method of Lagrange multipliers to determine the points on S that are the furthest away from the origin and the points on S that are the closest to the origin.

4. **[4+5+6=15 Points]** Let a, b and c be continuous functions $\mathbb{R} \to \mathbb{R}$.

(a) Show that

$$\mathbf{F} = (a(x)+(-1)^{n_1}y+(n_3+1)z)\,\mathbf{i}+((-1)^{n_1}x+b(y)+(n_2+1)z)\,\mathbf{j}+((n_3+1)x+(n_2+1)y+c(z))\,\mathbf{k}$$

is conservative.

(b) Determine a scalar potential for $\mathbf{F}$. Note that the potential function will involve integrals of a, b and c.

(c) For $a(x) = (-1)^{n_1}x^3$, $b(y) = (-1)^{n_2}y^2$ and $c(z) = (-1)^{n_3}z$, compute the line integral along the straight line segment connecting the point $p = \mathbf{i} - \mathbf{j}$ to the point $q = \mathbf{j} + \mathbf{k}$. Verify this result using the potential function from part (b).

5. **[15 Points]** Verify Gauss' Divergence Theorem for the vector field $\mathbf{F}$ in $\mathbb{R}^3$ defined as

$$\begin{aligned}\mathbf{F}(x,y,z) &= ((-1)^{n_1}x+(-1)^{n_2}y+(-1)^{n_3}z)\,\mathbf{i}+\\ &\quad ((-1)^{n_2}x+(-1)^{n_3}y+(-1)^{n_1}z)\,\mathbf{j}+\\ &\quad ((-1)^{n_3}x+(-1)^{n_1}y+(-1)^{n_2}z)\,\mathbf{k}\end{aligned}$$

and the cube $D = [-1,1]\times[-1,1]\times[-1,1]$.

6. **[7+8=15 Points]**

(a) Let $C \subset \mathbb{R}^2$ be the curve defined by the parametrization

$$\mathbf{r}(t) = (3\cos(t)+(-1)^{n_1}\cos(3t),\ 3\sin(t)+(-1)^{n_2}\sin(3t)),$$

with $t \in [0, 2\pi]$. Let $D \subset \mathbb{R}^2$ be the bounded region with boundary $\partial D = C$. Find the area of the set of those points $(x, y) \in D$ such that $(1+(-1)^{n_3})x+(1+(-1)^{1+n_3})y \geq 0$.

(b) Compute

$$\int_C f(x,y,z)\,\mathrm{d}x + g(x,y,z)\,\mathrm{d}y + h(x,y,z)\,\mathrm{d}z,$$

where

$$f(x,y,z) = x\left(\frac{n_1}{n_1x^2+n_2y^2}+\frac{n_3}{n_3x^2+n_4z^2}\right),$$

$$g(x,y,z) = y\left(\frac{n_2}{n_1x^2+n_2y^2}+\frac{n_5}{n_5y^2+n_6z^2}\right),$$

$$h(x,y,z) = z\left(\frac{n_4}{n_3x^2+n_4z^2}+\frac{n_6}{n_5y^2+n_6z^2}\right),$$

and C is the intersection of the surfaces $y^2+z^2 = n_7$ and $x+y+z = n_7$, oriented clockwise around the x-axis.

GOOD LUCK!

SC-S-C